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FUNDAMENTAL SOLUTION TO THE DIFFUSION BOUNDARY LAYER EQUATION FOR SEPARATED FLOW OVER SOLID SURFACES AT VERY LARGE PRANDTL NUMBERS†

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Abstract—Using the method of self-similar solutions an exact fundamental solution to the equation of convective diffusion is obtained for the case of separated flow over a surface when the Prandtl number is very large compared to unity. The driving force for diffusion is assumed to change abruptly from zero to unity at the streamwise location $x = \xi$ and an arbitrary distribution of the velocity profile curvature parameter $(\partial^2 u/\partial y^2)_{y=0}$ is allowed through the use of the von Mises transformation. An integral method using a cubic polynomial for the dimensionless concentration profile is shown to predict the correct functional form of the transfer coefficient but overestimates its magnitude by 3·3 per cent. This is more than twice the error of a corresponding integral method solution to the nonseparating flow problem, suggesting once again that for a given number of terms in the approximating polynomial the accuracy of profile methods in general deteriorates as one nears separation of the velocity boundary layer. The exact solution is specialized to the case of separated laminar wedge flow and transfer coefficients are derived for both step-change and power-law driving force distributions. These results have been used to compare the effect of pressure gradient (wedge variable) on the sensitivity of local transfer coefficients to streamwise gradients in driving force. In the limit $\xi \to 0$ results for the step change in driving force are shown to reduce to the proper "isothermal" coefficient at very large Prandtl numbers.

NOMENCLATURE

c, local concentration (mass fraction) of transferred species, equation (6);
D, Fick diffusion coefficient, equation (6);
f, nondimensional stream function, equation (49);
g, function in velocity profile law $u(x, y) = g(x) \cdot y^s$;
j'', rate of diffusional transfer per unit time and area, equation (7);
l, driving force gradient parameter, equation (58);
m, inviscid velocity gradient parameter, equation (49);
Nu, local Nusselt number based on distance x, equation (9);
p, parameter in equation (57);
Pr, Prandtl number ($= \nu/D$ for diffusion);
q, parameter in equation (57);
Re, Reynolds number;
s, exponent in power law $u(x, y) = g(x) \cdot y^s$;
u, x-component of fluid velocity;
v, y-component of fluid velocity;
v_* characteristic velocity, equation (4), or "friction" velocity;
x, distance in streamwise direction along solid surface; or argument of incomplete gamma function;
x^+, stretched streamwise distance, equation (16);
y, distance normal to solid surface.

Greek symbols

α, parameter in incomplete gamma function, equation (30);
β, wedge parameter, $2m/(1 + m)$;
$\Gamma(\alpha)$, gamma function of argument α;
$\gamma(\alpha, x)$, incomplete complementary gamma function of argument x with parameter α, equation (30);

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δ, diffusion boundary layer thickness, equation (39);
Δ_2, convection thickness, equation (36);
η, similarity variable, equation (23);
ϑ, concentration excess, $c - c_w$;
Θ, normalized concentration excess $(c - c_w)/(c_e - c_w)$;
λ, dummy (integration) variable or transformation factor;
μ, dynamic viscosity of fluid;
ν, kinematic viscosity of fluid, $= \mu/\rho$;
ξ, x-position of step change in driving force;
ρ, absolute (mass) density of fluid;
τ_w, shear stress, equation (1);
Ψ, stream function, equation (10);
Ψ^+, "stretched" stream function, equation (17).

Subscripts
e, at outer edge of boundary layer;
iso, corresponding to constant diffusional driving force;
w, at wall ($y = 0$);
x, based on distance x;
y, partial derivative with respect to y at constant x;
$y = 0$, at wall.

1. INTRODUCTION

WHEN treating forced convection problems in which the driving force for diffusion† or heat conduction varies from point to point along the surface, linearity of the convective diffusion equation enables the desired solution to be written down if a so-called "fundamental" solution is available [1, 2]. The latter solution is that pertaining to a physical situation in which the driving force for diffusion or heat transfer abruptly changes from zero to unity at some upstream point along the surface. While many such "step function" solutions must of necessity be obtained by approximate analytical methods, [3, 4], or directly from experiment [5, 6] several exact fundamental solutions can be obtained analytically using the method of similar solutions. This is particularly true for laminar flows at very large Prandtl numbers when the velocity profile within the diffusion boundary layer assumes a simple analytical form. Examples of such solutions are provided by the work of Lighthill [7] Acrivos [8] and Kestin and Persen [9]. These authors have treated large Prandtl number flows for which the streamwise velocity in the diffusion boundary layer may be accurately replaced by the linear law

$$u(x, y) = \frac{\tau_w(x)}{\mu} \cdot y \qquad (1)$$

where τ_w is the local shear stress, $\mu(\partial u/\partial y)_{y=0}$ at the surface. As pointed out by Spalding [10] such solutions in fact apply to turbulent boundary layer flows as well, since in the asymptotic extreme $Pr \to \infty$ the diffusion boundary layer is fully submerged within the laminar sublayer. However, it is well known that this representation of the velocity field breaks down as one approaches the condition of zero shear stress (near separation) since in this region, the next term in the Taylor series expansion

$$u(x, y) = \left(\frac{\partial u}{\partial y}\right)_{y=0} \cdot y + \frac{1}{2!}\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0} \cdot y^2 + \frac{1}{3!}\left(\frac{\partial^3 u}{\partial y^3}\right)_{y=0} \cdot y^3 + \ldots \qquad (2)$$

begins to dominate the first. When this is the case the very general and asymptotically exact fundamental solutions referred to above become inapplicable and one is led to inquire if an exact fundamental solution can be obtained in the nearly separated flow regime. While perhaps of less practical importance, solutions in this extreme are of great theoretical interest since, being exact, they supply test cases and useful asymptotes to which certain related problems in heat- and mass-transfer theory must conform. Moreover, closed form exact solutions are valuable in that they reveal functional dependences frequently obscured in available numerical solutions. With this in mind we have investigated the

† Throughout the present paper reference to diffusion should not be taken to imply *net* transfer across the fluid/solid interface. While the following discussion is readily extended to this case, the solutions presented strictly apply to the case of negligible interfacial mass velocity, as encountered, for example, at the surface of impermeable catalytic solids in flow systems, or simply when the *net* mass-transfer rate is sufficiently small.

extreme in which the velocity field near the surface can be well represented by the quadratic term of (2), viz.

$$u(x, y) \approx \frac{1}{2}\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0} \cdot y^2. \tag{3}$$

This case has been dealt with summarily by Acrivos [8], who has provided transfer coefficient expressions for velocity profiles of the general form $u(x, y) = g(x) \cdot y^s$. In the present paper the important special case $s = 2$ is explored in greater detail, with emphasis on the similarities which exist between the separated and nonseparated flow solutions. Thus, the discussion of Section 2 will instead parallel Kestin and Persen's [9] treatment of the $s = 1$ case, and, with suitably defined variables, it will be shown that the results can be cast in a form analagous to those governing the nonseparated flow problem. In this regard a particularly useful variable is the "characteristic velocity" distribution $v_*(x)$ defined by

$$v_*(x) \equiv \left[\frac{\nu^2}{2}\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0}\right]^{1/3}. \tag{4}$$

As will be seen $v_*(x)$ plays a role similar to the familiar "friction" velocity $v_*(x) \equiv [\tau_w(x)/\rho]^{1/2}$ in the nonseparated flow problem.

A closed form solution is first given for the case in which the characteristic velocity $v_*(x)$ may have an arbitrary distribution along the surface, and in Section 4 this result is specialized to the case of a laminar wedge flow with both step function, and power law driving force distributions. When the step change occurs at the leading edge of the surface ($\xi = 0$) we recover the asymptotically exact Nusselt number result

$$Nu_{iso}/\sqrt{(Re_x)} = 0{\cdot}224468\ Pr^{1/4} \tag{5}$$

in agreement with the accurate tabular solutions recently given by Evans [11] for the extreme of very large Prandtl number. In Section 3 the exact solution for arbitrary $v_*(x)$ is compared with an integral method solution of the same problem, and the results are compared with a similar set of calculations for the nonseparated flow case.

2. ANALYSIS AND EXACT SOLUTION

We consider the physical configuration sketched in Fig. 1 and adopt as our starting point the constant property steady state diffusion equation for laminar boundary layer flow in two dimensions, viz.

$$u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} = D\frac{\partial^2 c}{\partial y^2} \tag{6}$$

where c represents the mass fraction of a trace component present in the carrier fluid.† In regions where the diffusional driving force $\vartheta_e \equiv c_e - c_w$ is constant the normalized concentration excess $\Theta \equiv \vartheta/\vartheta_e$ will satisfy this same equation, where the coefficient D represents the

† As is well known, the diffusion of heat is governed by mathematically identical laws. The solutions given herein therefore apply equally well to the calculation of heat transfer in separated flow at very large Prandtl numbers. This fact has influenced our subsequent choice of nomenclature.

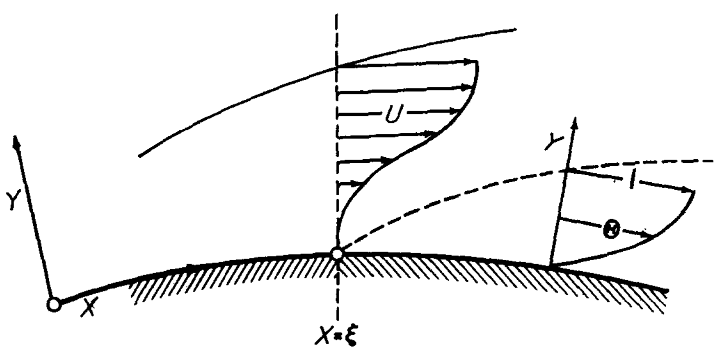

FIG. 1. System co-ordinates and notation.

pertinent diffusion coefficient. At the streamwise station $x = \xi$ the driving force $\vartheta_e \equiv c_e - c_w$ abruptly changes from the value zero to unity and remains constant thereafter, so that the diffusion boundary layer (shown dashed) itself grows under the influence of a constant driving force ϑ_e = constant = 1 (for $x > \xi$). Let us now focus our attention on the corresponding rate of transfer, $-j''(x, 0; \xi)$, to the surface downstream of the step change and therefore examine, in the absence of *net* mass transfer, the normal gradient $\partial\Theta/\partial y$ evaluated at $y = 0$. In what follows this gradient will be written $\Theta_y(x, 0; \xi)$ as a reminder of its dependence on both the position x and the upstream location of the step change. Once one has obtained the fundamental solution $\Theta_y(x, 0; \xi)$ transfer rates can be computed from the linear law

$$-j''(x, 0; \xi) = D\rho\vartheta_e \cdot \Theta_y(x, 0; \xi) \tag{7}$$

where, in this instance, $\vartheta_e = 1$. As discussed by Rubesin [1], Tribus and Klein, [2] *et al.*, transfer rates for arbitrary distributions of driving force $\vartheta_e(x)$ can then be calculated from the superposition law

$$-j''(x, 0) = D\rho \int_{\xi=0}^{x} \Theta_y(x, 0; \xi) \cdot \mathrm{d}\vartheta_e(\xi) \tag{8}$$

where the fundamental solution discussed in the present paper appears as a kernel and integration is taken in the Stieltjes sense. It should be observed that the fundamental solution may also be presented in the form of a Nusselt number based on the distance x; viz.

$$Nu_x \equiv \frac{-j''(x, 0; \xi)}{D\rho\vartheta_e/x} = x \cdot \Theta_y(x, 0; \xi). \tag{9}$$

When $\xi \to 0$ we should therefore recover the "*iso*-compositional" (constant driving force) value of the Nusselt number, abbreviated hereafter as Nu_{iso}.

As in the case of nonseparated flows the present problem may be solved by use of the method of similar solutions. This technique, suggested by the absence of a characteristic length governing the development of the diffusion boundary layer allows the problem to be reduced to the solution of a simple ordinary differential equation. One thus anticipates a "universal" profile $\Theta(\eta)$ from which the fundamental solution $\Theta_y(x, 0; \xi)$ can be calculated using the value of $\Theta'(0)$ and the transformation properties of the appropriate similarity variable $\eta(x, y)$.

This may be carried out as follows. First the convective diffusion equation is reduced to the form of a variable property transient "conduction" equation by invoking the von Mises transformation, i.e. by replacing the independent variable y with the stream function $\Psi(x, y)$, defined here by the relations†

$$u = \nu \frac{\partial\Psi}{\partial y}, \quad v = -\nu \frac{\partial\Psi}{\partial x}. \tag{10}$$

In this way one obtains the partial differential equation

$$\frac{\partial\Theta}{\partial x} = \frac{1}{Pr} \cdot \frac{\partial}{\partial\Psi}\left(\frac{u}{\nu}\frac{\partial\Theta}{\partial\Psi}\right) \tag{11}$$

where Pr represents the diffusional Prandtl number‡ ν/D and the operator $\partial/\partial x$ now implies partial differentiation with respect to x, *holding the stream function* Ψ constant. At this point the local velocity u appearing on the right-hand side of (11) must be expressed in terms of x and Ψ alone, making use of the quadratic law [cf. (3)] for nearly separated flows at large Prandtl numbers. In terms of the characteristic velocity $v_*(x)$ defined above this profile law may be cast in the form

$$u = v_*(x) \left[\frac{v_*(x)\, y}{\nu}\right]^2 \tag{12}$$

which, in turn, implies the stream function dependence

$$\Psi(x, y) = \frac{1}{3}\left[\frac{v_*(x)\, y}{\nu}\right]^3. \tag{13}$$

Therefore in (11) the local velocity $u(x, \Psi)$ can be expressed

$$u(x, \Psi) = v_*(x) \cdot (3\Psi)^{2/3}. \tag{14}$$

† The factor ν (kinematic viscosity) is introduced in the above equations to render the stream function Ψ dimensionless.

‡ In the Western literature this property group is usually called the Schmidt number. This writer prefers the usage common in the Russian mass transfer literature since it does not obscure an obvious analogy. Similarly, we have used the symbol Nu in place of the equivalent Sherwood number.

Inserting this expression into (11) yields the partial differential equation

$$\frac{\nu}{v_*(x)}\frac{\partial\Theta}{\partial x} = \frac{1}{Pr}\cdot\frac{\partial}{\partial\Psi}\left[(3\Psi)^{2/3}\frac{\partial\Theta}{\partial\Psi}\right]. \quad (15)$$

Inspection of this result reveals that considerable formal simplification is possible by introducing the new "stretched" independent variables

$$x^+ \equiv \int_\xi^x \frac{v_*(x)\cdot \mathrm{d}x}{\nu} \quad (16)$$

$$\Psi^+ \equiv 3^{-1/2}\, Pr^{3/4}\, \Psi. \quad (17)$$

Doing this we are left with the partial differential equation

$$\frac{\partial\Theta}{\partial x^+} = \frac{\partial}{\partial\Psi^+}\left[(\Psi^+)^{2/3}\frac{\partial\Theta}{\partial\Psi^+}\right] \quad (18)$$

which must be solved subject to the boundary conditions

$$\Theta = 1 \text{ for } x^+ = 0\ ;\ \text{all } \Psi^+ \geqslant 0 \quad (19)$$

$$\Theta = 1 \text{ for } \Psi^+ = \infty\ ;\ \text{all } x^+ \geqslant 0 \quad (20)$$

$$\Theta = 0 \text{ for } \Psi^+ = 0\ ;\ \text{all } x^+ \geqslant 0. \quad (21)$$

As anticipated, a solution to the above boundary value problem can be constructed such that the x^+ and Ψ^+ dependence of the Θ-field is contained in single independent variable η where a satisfactory choice of η is found to be†

$$\eta \equiv (9/16)^{1/4}(x^+)^{-1/4}(\Psi^+)^{1/3}. \quad (22)$$

In terms of the original independent variables of the problem this is equivalent to the definition

$$\eta = \frac{1}{2}\,\frac{v_*(x)\cdot y}{\nu}\cdot Pr^{1/4}\left[\int_\xi^x \frac{v_*(x)\cdot \mathrm{d}x}{\nu}\right]^{1/4}. \quad (23)$$

With any choice of this general form the boundary conditions at $x^+ = 0$ and $\Psi^+ = \infty$ become boundary conditions at $\eta = \infty$ i.e. (19, 20, 21) are replaced by the two statements

$$\Theta(\infty) = 1, \quad (24)$$

$$\Theta(0) = 0. \quad (25)$$

That a solution of the form $\Theta(\eta)$ exists may be verified by subjecting (18) to the further co-ordinate transformation $(x^+, \Psi^+) \to (x^+, \eta)$. In this way one finds that $\Theta(x^+, \eta)$ must satisfy the equation

$$\frac{1}{4\eta^2}\frac{\partial^2\Theta}{\partial\eta^2} + \eta\frac{\partial\Theta}{\partial\eta} = 4x^+\frac{\partial\Theta}{\partial x^+} \quad (26)$$

where the operator $\partial/\partial x^+$ now implies partial differentiation with respect to x^+ *at constant* η. Inspection of (26) reveals that a solution of the form $\Theta(\eta)$ indeed exists, where $\Theta(\eta)$ must satisfy the linear second order *ordinary* differential equation

$$\Theta'' + 4\eta^3\Theta' = 0. \quad (27)$$

This equation is first order in the first derivative Θ' and its solution is readily written in terms of well tabulated functions. Integrating twice, one obtains the expression

$$\Theta(\eta) = \frac{\int_0^\eta \exp(-\eta^4)\cdot \mathrm{d}\eta}{\int_0^\infty \exp(-\eta^4)\cdot \mathrm{d}\eta}. \quad (28)$$

For diffusional transfer calculations the derivative Θ' evaluated at $\eta = 0$ is of interest. Equation (28) implies

$$\Theta'(0) = [\int_0^\infty \exp(-\eta^4)\cdot \mathrm{d}\eta]^{-1}. \quad (29)$$

The indefinite integrals appearing above are closely related to one of the incomplete gamma functions [12, 13]

$$\gamma(\alpha, x) \equiv \int_0^x \exp(-\lambda)\cdot\lambda^{\alpha-1}\,\mathrm{d}\lambda. \quad (30)$$

Tabular values of this function are provided in the work of Pearson [13]. In terms of $\gamma(\alpha, x)$ the universal profile $\Theta(\eta)$ may be expressed

$$\Theta(\eta) = \frac{\gamma(1/4, \eta^4)}{\gamma(1/4, \infty)} \quad (31)$$

where $\gamma(1/4, \infty) = \Gamma(1/4) = 3{\cdot}62560991$ [14]. A graph of this function is given in Fig. 2 and compared with the function $\gamma(1/3, \eta^3)/\gamma(1/3, \infty)$ which arises in the theory of nonseparated boundary

† This property can be arrived at by noticing that (18) is invariant under the co-ordinate transformation $x^+ \to \lambda x^+$, $\Psi^+ \to \lambda^{3/4}\,\Psi^+$. In view of this fact and the form of the boundary conditions (19, 20, 21) the *solution* $\Theta(x^+, \Psi^+)$ must also be invariant under this transformation. This condition will certainly be met if Θ depends only upon a combination of the independent variables x^+ and Ψ^+ which is itself invariant under this transformation. Equation (22) represents one such combination.

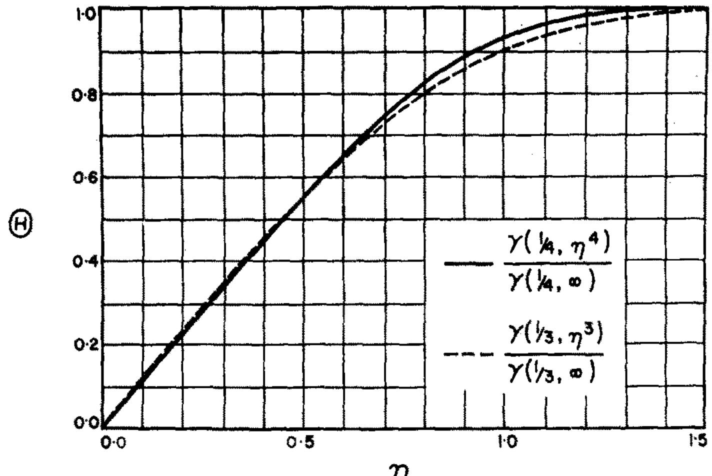

FIG. 2. Universal diffusion boundary layer profiles.

layers† [9]. For small values of η we have the expansion

$$\Theta(\eta) = \frac{4}{\Gamma(1/4)} \cdot \eta \left\{ 1 - \frac{1}{5}\eta^4 + \frac{1}{18}\eta^8 - \frac{1}{78}\eta^{12} + \frac{1}{408}\eta^{16} - \dots \right\} \quad (32)$$

so that $\Theta'(0) = 4/\Gamma(1/4) = 1{\cdot}103262652$. It is then possible to find $\Theta_y(x, 0; \xi)$ from $\Theta'(0)$ and the transformation properties of the similarity variable, since

$$\Theta_y(x, 0; \xi) \equiv \left(\frac{\partial \Theta}{\partial y}\right)_{y=0} = \left(\frac{d\Theta}{d\eta}\right)_{\eta=0} \left(\frac{\partial \eta}{\partial y}\right)_{y=0}. \quad (33)$$

Carrying out the indicated operations we find

$$\Theta_y(x, 0; \xi) = \frac{2}{\Gamma(1/4)} Pr^{1/4} \frac{v_*(x)}{\nu} \left[\int_\xi^x \frac{v_*(x) \cdot dx}{\nu} \right]^{-1/4} \quad (34)$$

which constitutes the relation sought. This result should be compared with its counterpart for nonseparated flow at large Prandtl numbers, viz. [9]

$$\Theta_y(x, 0; \xi) = \frac{3^{1/3}}{\Gamma(2/3)} Pr^{1/3} \frac{v_*(x)}{\nu} \cdot \left[\int_\xi^x \frac{v_*(x) \cdot dx}{\nu} \right]^{-1/3}. \quad (35)$$

It should be remembered that in the latter case the characteristic velocity $v_*(x)$ is the "friction velocity" $[\tau_w(x)/\rho]^{1/2}$ whereas in the former $v_*(x)$ is given by (4). It is seen that in terms of these respective characteristic velocities the fundamental solutions are strikingly similar in structure.

From a theoretical point of view application of (34) to the well studied case of separated laminar wedge flow immediately comes to mind. Such an application not only provides results of greater generality than those available in the form of existing similar solutions, it also provides a valuable check on the mathematical steps preceding (34). For example, in the limit $\xi \to 0$ we should be able to recover the "isothermal" value $Nu_{x,iso}$ of the local Nusselt number for separated wedge flow at very large Prandtl numbers. However, before discussing this class of applications, we digress for a moment to illustrate the applicability of the well-known integral (profile) method to the problem already treated exactly in this section.

† For this problem Kestin and Persen [9] used a similarity variable which is proportional to the cube of our variable η. This accounts for the infinite value of $\Theta'(0)$ in their Fig. 2, which is removed in their calculation of Θ_y by the corresponding singularity of $(\partial\eta/\partial y)_{y=0}$.

3. COMPARISON WITH AN APPROXIMATE FUNDAMENTAL SOLUTION; THE INTEGRAL METHOD

Having an exact fundamental solution for the case of separating flow it is of interest to examine the success of integral methods when applied to the same problem. For instance, it is known that the von Kármán–Pohlhausen approach with a cubic polynomial Θ-profile predicts the non-separating flow fundamental solution for $Pr \gg 1$ to within 1·5 per cent of the exact value [15], and correctly represents its functional form. We therefore anticipate that integral methods will be able to reproduce the functional form of (34) however, it remains to be seen how well they will predict the absolute value of the multiplicative constant.

Integration of (6) from $y = 0$ to $y = \infty$ provides a relation for the growth of the "convection" thickness Δ_2 defined by

$$\Delta_2 \equiv \int_0^\infty \frac{u}{u_e}[1 - \Theta] \cdot \mathrm{d}y. \tag{36}$$

This integral relation for the boundary layer can be cast in the general form (e.g. [16])

$$\frac{u_e}{\nu}\frac{\mathrm{d}\Delta_2^2}{\mathrm{d}x} = \frac{2}{Pr}\left(\frac{\partial\Theta}{\partial y}\right)_{y=0}\Delta_2 - 2\frac{\Delta_2^2}{\nu}\frac{\mathrm{d}u_e}{\mathrm{d}x} - 2\frac{u_e\Delta_2^2}{\nu}\frac{\mathrm{d}}{\mathrm{d}x}(\ln \vartheta_e). \tag{37}$$

In applying this relation to the present problem the diffusional driving force is constant (at the value unity) for $x > \xi$ so that the last term of (37) can be dropped. Making use of the definition of the local Nusselt number the simplified integral equation may be written in the compact form

$$Nu = x \cdot Pr \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u_e\Delta_2}{\nu}\right). \tag{38}$$

In accordance with the von Kármán–Pohlhausen procedure we first introduce a functional form for the profile $\Theta(y/\delta)$ where δ represents the yet undetermined thickness of the diffusion boundary layer. Regardless of the choice of functional form it is dimensionally clear that $\Theta_y(0) \propto \delta^{-1}$. The integral equation (38) then becomes an ordinary differential equation for the growth of the diffusion boundary layer thickness $\delta(x; \xi)$ downstream of the point $x = \xi$ or, alternatively, a differential equation for the decay of the non-dimensional transfer coefficient $Nu(x; \xi)$. This will be illustrated here for the case in which the local Θ-field is represented by a cubic polynomial, viz.

$$\Theta = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 \quad \text{for } 0 \leqslant y/\delta \leqslant 1. \tag{39}$$

This functional form is simple to work with, has zero slope at $y/\delta = 1$, and satisfies the two most important conditions of the problem, i.e.

$$\Theta(x, 0; \xi) = 0 \tag{40}$$

$$\Theta(x, \delta; \xi) = 1. \tag{41}$$

It is clear from this choice that $\Theta_y(x, 0; \xi)$ and $\delta(x; \xi)$ are related by

$$\Theta_y(x, 0; \xi) = \frac{3}{2} \cdot \frac{1}{\delta} \tag{42}$$

or, equivalently [cf. (9)]

$$Nu(x; \xi) = \frac{3}{2} \cdot \frac{x}{\delta}. \tag{43}$$

To parallel the development of the preceding section we assume the diffusion layer grows in a region within which the velocity profile $u(x, y)$ is well represented by the quadratic law given by (12). Then, using its definition, the "convection thickness" Δ_2 may be evaluated as

$$\Delta_2 = \delta \cdot \frac{v_*}{u_e}\left(\frac{v_*\delta}{\nu}\right)^2 \int_0^1 \lambda^2\left[1 - \left(\frac{3}{2}\lambda - \frac{1}{2}\lambda^3\right)\right]\mathrm{d}\lambda. \tag{44}$$

Performing the indicated operations and eliminating the thickness δ in favor of the nondimensional transfer coefficient, Nu, we find

$$\Delta_2 = \frac{27}{192}\frac{v_*}{u_e} \cdot \left(\frac{v_* x}{\nu}\right)^2 \cdot \frac{x}{Nu^3}. \tag{45}$$

Therefore, the integral equation, (38), can be written

$$Nu = x \cdot Pr \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{27}{192}\left(\frac{v_* x}{\nu}\right)^3 \cdot \frac{1}{Nu^3}\right]. \tag{46}$$

Differentiation and rearrangement yield the following Bernoulli equation for $Nu(x;\,\xi)$

$$\frac{\mathrm{d}}{\mathrm{d}x} Nu - \left[\frac{\mathrm{d}}{\mathrm{d}x} \ln\left(\frac{v_* x}{\nu}\right)\right] Nu = -\frac{1}{3}\left(\frac{192}{27}\right)\frac{1}{Pr}\left(\frac{v_* x}{\nu}\right)^{-3}\frac{1}{x}\cdot Nu^5. \quad (47)$$

Therefore, this equation becomes linear in the dependent variable $(Nu)^{-4}$, having the integrating factor $(v_* x/\nu)^4$. Subject to the condition $Nu(\xi;\,\xi) = \infty$ this linear differential equation can be integrated and solved for $Nu(x;\,\xi)$. There results

$$Nu = \left[\frac{3}{4}\left(\frac{27}{192}\right)\right]^{1/4} Pr^{1/4}\cdot\frac{v_*(x)x}{\nu}\left[\int_\xi^x \frac{v_*(x)\cdot \mathrm{d}x}{\nu}\right]^{-1/4}. \quad (48)$$

Keeping in mind (9), one notices that (48) is identical in form to (34), differing only in the value of the multiplicative constant. Evaluation of the fourth root shown in (48) yields 0·5698767642 as compared with $2/\Gamma(1/4) =$ 0·551631326. Thus the integral method using a cubic polynomial profile reproduces the functional form of the exact solution but overestimates absolute values by some 3·3 per cent. This error is more than twice that obtained in the nonseparated flow case, suggesting once again that for a given number of terms in the approximating polynomials the accuracy of profile methods will in general deteriorate as one approaches the condition of separation.

4. SPECIALIZATION TO SEPARATED WEDGE FLOW

We return now to the exact fundamental solution and its specialization to the case of separated laminar wedge flow. Separated wedge flow is a special case of the class of similar solutions to the steady flow laminar boundary layer equations for free stream velocity profiles of the power law family $u_e \propto x^m$ where m is a constant. For this family exact solutions to the velocity field are available in terms of a well tabulated nondimensional stream function† f and its derivatives [17, 18], where

$$f \equiv \frac{\nu\Psi}{u_e}\cdot\left(\frac{1}{\nu\beta}\,\frac{\mathrm{d}u_e}{\mathrm{d}x}\right)^{1/2} \quad (49)$$

is a function of the similarity variable

$$y\cdot\left(\frac{1}{\nu\beta}\,\frac{\mathrm{d}u_e}{\mathrm{d}x}\right)^{1/2}. \quad (50)$$

The parameter β appearing in these transformations is a "wedge parameter" related to the exponent m by $\beta \equiv 2m/(1+m)$. For the singular value $\beta = -0{\cdot}198838$ (corresponding to $m = -0{\cdot}09042867$) the laminar velocity boundary layer is separated ($\tau_w = 0$) everywhere. Since the normalized fluid velocity u/u_e is simply given by f', where the prime denotes differentiation with respect to the similarity variable (50), separation corresponds to the case $f''(0) = 0$. From the definition of the characteristic velocity $v_*(x)$ and the similarity variables above it is seen that $v_*(x)$ is related to the corresponding magnitude of $f'''(0)$; indeed, for the case of separated wedge flow we explicitly find

$$v_*(x) = u_e(x)\cdot\left[\frac{1}{2\cdot(2-\beta)}\cdot Re_x^{-1}\cdot f'''(0)\right]^{1/3} \quad (51)$$

where $\beta = -0{\cdot}198838$ and $f'''(0) = 0{\cdot}198838$ [18].

4.1 *Step change in driving force at $x = \xi$*

Inserting this $v_*(x)$ distribution into the general expression, (34), gives the explicit Nusselt number result:

$$\frac{Nu}{\sqrt{Re_x}} = \frac{1}{\sqrt{(2-\beta)}}\cdot\frac{4}{\Gamma(1/4)}\left[\frac{f'''(0)}{4!}\right]^{1/4} Pr^{1/4}\left[1-\left(\frac{\xi}{x}\right)^{(2/3)(m+1)}\right]^{-1/4}. \quad (52)$$

If one now introduces the appropriate numerical values of β, m and $f'''(0)$ for separated laminar wedge flow (52) reduces to

† The stream function Ψ appearing in (49) is that defined by (10). The product $\nu\Psi$ may be identified with the stream function given, for example, by Evans [11] and Spalding [18].

$$Nu/\sqrt{Re_x} = 0{\cdot}224468\ Pr^{1/4}[1 - (\xi/x)^{0{\cdot}606301}]^{-1/4} \quad (53)$$

which reduces further to the form of equation (5) when $\xi \to 0$. For this special case the results given above are in agreement with the accurate solutions recently reported by Evans [11]. The latter work further suggests that in practice (52) may be extended to lower Prandtl numbers by replacing $\Gamma(1/4)$ with the asymptotic series

$$\Gamma(1/4) + \frac{0{\cdot}0840531}{Pr} - \frac{0{\cdot}00590720}{Pr^2} - \frac{0{\cdot}000105923}{Pr^3} + 0(Pr^{-4}). \quad (54)$$

4.2. *Power law driving force distribution*

The superposition law expressed by (8) can be combined with the results of the previous section to derive an expression for the effect of power law driving force distributions on the local transfer coefficient. Thus we imagine that $\vartheta_e(x)$ obeys the power law $\vartheta_e(x) \propto x^l$ where l is some positive or negative constant. Introducing the following kernel into (8)

$$\Theta_y(x, 0; \xi) = Nu_{iso} \cdot \frac{1}{x}\left[1 - \left(\frac{\xi}{x}\right)^{(2/3)(m+1)}\right]^{-1/4} \quad (55)$$

one thereby obtains the result

$$-j''(x) = Nu_{iso}\frac{D\rho\vartheta_e(x)}{x}\ l \int_0^1 [1 - \lambda^{(2/3)(m+1)}]^{-1/4}\ \lambda^{l-1}\,d\lambda. \quad (56)$$

The integral appearing here can be evaluated in terms of the Gauss factorial function with the help of the relations [12]

$$\int_0^1 \lambda^{p-1}(1-\lambda)^{q-1}\,d\lambda = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \frac{(p-1)!\,(q-1)!}{(p+q-1)!} \quad (57)$$

where $p, q > 0$.

Carrying this procedure through, the result for the corresponding transfer coefficient ratio can be written

$$\frac{Nu}{Nu_{iso}} = \frac{\left[\dfrac{l}{(2/3)(m+1)}\right]!\left[-\dfrac{1}{4}\right]!}{\left[\dfrac{l}{(2/3)(m+1)} - \dfrac{1}{4}\right]!} \quad (58)$$

where $m = -0{\cdot}09042867$. It is of interest to compare this result with its counterpart for non-separated wedge flows. In the latter case it can be shown that

$$\frac{Nu}{Nu_{iso}} = \frac{\left[\dfrac{l}{(3/4)(m+1)}\right]!\left[-\dfrac{1}{3}\right]!}{\left[\dfrac{l}{(3/4)(m+1)} - \dfrac{1}{3}\right]!}. \quad (59)$$

These explicit relations can be conveniently used to evaluate the effect of the pressure gradient (wedge) parameter m on the sensitivity of local transfer coefficients to streamwise gradients in diffusional driving force. Results are summarized in Table 1 and Fig. 3. Also included for comparison are the numerical results of Levy [19] calculated by a finite difference method for the case $Pr = 10$. It is seen that small pressure gradient flows are much more sensitive to steamwise gradients in diffusional driving force than strongly accelerated flows. For instance one can compare the magnitudes of l required to cause the local heat flux to vanish, i.e. to "separate" the Θ-profile, despite the presence of a non-zero local driving force. This is done by noting when the argument of the factorial functions in the denominators of (58) and (59) take on the values -1, since $(-1)! = \infty$. Regardless of which formula is used one obtains $Nu/Nu_{iso} = 0$ for $l = -(1/2)(m+1) = -(2-\beta)^{-1}$ in agreement with the results of Levy [19]. Since the requisite gradient parameter l is observed to become larger in absolute value as m increases, strongly accelerating boundary layer flows (large m) are far less sensitive to streamwise gradients in driving force than are small pressure gradient flows ($m \approx 0$). In the separated flow case (58) reveals that the local transfer coefficient will vanish when the parameter l takes on the value $-0{\cdot}454786$. This should be contrasted with the case of plane stagnation flow ($m = 1$) for which the value $l = -1$ is necessary to cause the local transfer coefficient to vanish.

Table 1. Calculated values of Nu/Nu_{iso}; comparison of analytical results for $Pr \gg 1$ with Levy's numerical results† for $Pr = 10$

l	$\beta = -0{\cdot}198838$		$\beta = 0$		$\beta = 1$	
	Equation (58)	Levy†	Equation (59)	Levy†	Equation (59)	Levy†
−1·00					0·0000	0·0000
−0·75					0·4312	0·3159
−0·50			0·0000	0·0000	0·6845	0·6294
−0·25	0·7040	0·6707	0·6845	0·6344	0·8624	0·8406
0·00	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000
0·25	1·1693	1·1795	1·2092	1·2268	1·1129	1·1283
0·50	1·2906	1·3057	1·3689	1·3961	1·2092	1·2364
0·75	1·3856		1·5000		1·2937	
1·00	1·4649	1·4862	1·6123	1·6476	1·3689	1·4137
1·50	1·5940		1·8000		1·5000	
2·00	1·6978	1·7258	1·9556	1·9923	1·6123	1·6788
3·00	1·8616	1·8929	2·2091	2·2417	1·8000	1·8802
4·00	1·9910	2·0255	2·4151	2·4394	1·9556	2·0451

† Calculated for $Pr = 10$ from four place tables given by Levy [19].

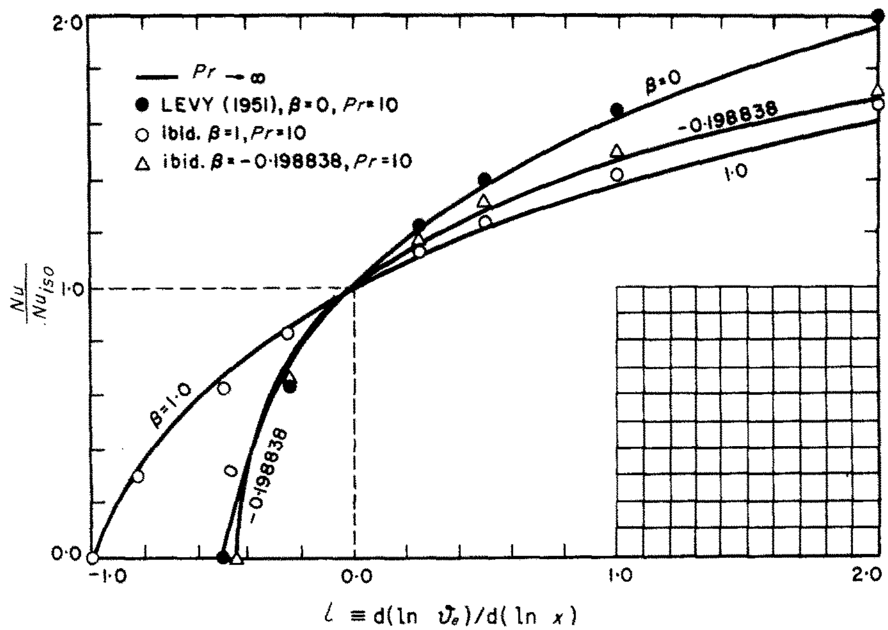

FIG. 3. Sensitivity of transfer coefficient to streamwise gradients in diffusional driving force for three values of the Hartree wedge (pressure gradient) parameter.

5. CONCLUDING REMARKS

By analogy with the corresponding non-separated flow solution, two remarks should be made concerning the restrictions placed on the present solution. The restriction to very large Prandtl number insures that the concentration diffusion boundary layer will remain thin compared with the velocity boundary layer thereby justifying the use of a simple velocity profile, (3), within the diffusion layer. If the Prandtl number is not large the solution is still valid for sufficiently small values of x^+ [cf. (16)] since the diffusion layer is certainly "thin" initially. Thus exact solutions for arbitrary Prandtl number should tend to the present solution for sufficiently small x^+.

Secondly, in the extreme $Pr \to \infty$ *or* $x^+ \to 0$ the solution may also be applicable to the case of turbulent flow within the velocity boundary layer provided laminar sublayers exist for separated turbulent boundary layer profiles. If this is the case then any general solution for the tubulent case should become asymptotic to (34) for sufficiently small values of x^+.

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Résumé—En utilisant la méthode des solutions "à similitude interne" on établit une solution fondamentale exacte de l'équation de diffusion convective dans le cas d'un écoulement sur une surface, avec nombre de Prandtl très supérieur à l'unité. On suppose que les forces de diffusion varient brutalement à partir de zéro, à l'abscisse $x = \xi$, la transformation de von Mises permet de tenir compte d'une distribution arbitraire du paramètre de courbure du profil de vitesses $(\partial^2 u/\partial y^2)_{y=0}$. Une méthode intégrale, utilisant une forme polynomiale du 3° ordre pour le profil de concentration sans dimensions, aboutit à une forme correcte du coefficient d'échange mais donne des valeurs trop élevées de 3,3%, c'est-à-dire plus de deux fois l'erreur d'une solution donnée par la méthode intégrale dans le cas d'un écoulement non décollé. Ceci suggère une fois de plus que, pour un nombre donné de termes dans l'approximation polynomiale, la précision des méthodes de profils baisse au voisinage du décollement de la couche limite.

Zusammenfassung—Mit Hilfe der Methode ähnlicher Lösungen erhält man für sehr grosse Prandtlzahlen eine exakte Fundamentallösung der Gleichung der konvektiven Diffusion für Oberflächenströmung in der Nähe der Ablösung. Die treibende Kraft der Diffusion soll sich plötzlich von Null an der Stelle $x = \xi$ in Strömungsrichtung ändern, wobei durch Benützung der von Mises-Transformation eine beliebige Verteilung des Krümmungsparameters des Geschwindigkeitsprofils $(\partial^2 u/\partial y^2)_{y=0}$ zulässig ist. Eine Integralmethode, die dem dimensionslosen Konzentrationsprofil ein kubisches Polynom zugrundelegt, führt auf die korrekte Funktionsform des Wärmeübergangskoeffizienten, gibt aber seine Grösse um 3,3 Prozent zu hoch an. Dieser Fehler ist mehr als doppelt so gross als der einer Lösung nach einer entsprechenden Integralmethode für die Strömung ohne Ablösung und zeigt wieder einmal, dass für eine gegebene Anzahl von Gliedern im Näherungspolynom die Genauigkeit der Profilmethoden im allgemeinen abnimmt bei Annäherung an die Ablösung. Die exakte Lösung beschränkt sich auf den Fall laminarer Keilströmung in der Nähe der Ablösung; die Übergangskoeffizienten sind sowohl für schrottweise Änderung als auch für Verteilung der treibenden Kraft nach einem Potenzgesetz abgeleitet. Mit diesen Ergebnissen liess sich der Einfluss des Druckgradienten (Keilvariable) auf die Empfindlichkeit des örtlichen Übergangskoeffizienten mit den Gradienten der treibenden Kraft in Strömungsrichtung vergleichen. Im Grenzfall sind die Ergebnisse für schrittweise Änderung der treibenden Kraft auf den entsprechenden "isothermen" Koeffizienten für sehr grosse Prandtlzahlen reduzierbar.

Аннотация—Получено с помощью метода подобных решений точное фундаментальное решение уравнения для конвективной диффузии для случая отрывного течения на поверхности при числе Прандтля значительно превышающем единицу. Предполагается, что величина безразмерной концентрации внезапно изменяется в точке $x = \xi$ от нуля до единицы и что распределение параметра $(\partial^2 u/\partial y^2)_{y=0}$ кривизны профиля скорости определяется с помощью преобразования фон Мизеса. Показано, что интегральный метод, использующий кубический полином для безразмерного профиля концентрации, определяет правильную функциональную форму коэффициента массообмена, но завышает его величину на 3,3%. Это в два раза превосходит ошибку соответствующего решения с помощью интегрального метода для задачи безотрывного течения. При этом принимается, что для данного числа членов аппроксимирующего полинома точность указанных методов, в общем, ухудшается вблизи отрыа скоростного пограничного слоя.

Точное решение применено к случаю отрывного обтекания клина ламинарным потоком. Коэффициенты переноса выведены как для ступенчатого распределения движущей силы, так и для степенного закона ее изменения. Результаты использовались для сравнения влияния градиента давления (раствор клина) на чувствительность локальных коэффициентов переноса к градиентам движущей силы (безразмерной концентрации). В заключении показано, что ступенчатое изменение движущей силы уменьшает истинный «изотермический» коэффициент переноса при весьма больших числах Прандтля.